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WAVE PROPAGATION IN HETEROGENEOUS ANISOTROPIC PLATES INVOLVING LARGE DEFLECTION

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Abstract—Equations of motion for laminated plates of composite materials are derived for motions of large amplitudes as well as for incremental deformations superposed on large deflections. Free waves of large and small amplitudes propagating, respectively, in initially flat and deformed plates are investigated. By using a special substitution for the extensional displacement components the governing equations appear to be linear and consequently the analysis of free wave propagation greatly simplified. It is found that a train of free wave consists of a finite number of simple harmonic waves.

1. INTRODUCTION

The problem of laminated plates subjected to impact of foreign objects has drawn great attention recently[1–3]. The interest resulted from the increasing use of laminated composites in the turbine blades of the jet engine, which are vulnerable to the impact of starlings or icicles if sucked into the engine. In the previous works, small deflection theories of plates were employed. However, if the impact would result in any fracture in the laminates, it would also be intense enough to cause large deflections in the plate. Accordingly, it would be more adequate to use the nonlinear plate theory to investigate propagation of such disturbances.

There have been papers on nonlinear dynamic analysis of laminated plates. All of them were dealing with vibrations of rectangular plates. Whitney and Leissa[4] derived a set of equations for laminated plates using the classical Kirchhoff hypotheses in conjunction with the nonlinear assumptions in the von Karman sense. Wu and Vinson[5] investigated the problem of nonlinear oscillations of an orthotropic plate including rotatory inertia and shear deformation effects. Their results indicated that the effect of shear deformation could be significant even in the range of moderately large length to thickness ratio. However, the formulation given by[5] did not include the strong bending–stretching coupling which could exist in laminates. Bennett[6] presented a study on large amplitude oscillations of a simply supported angle-ply laminate with the layers stacked in an arbitrary manner. More recently, Bert[7] has presented a simplified approach for the analysis of nonlinear vibrations of an arbitrarily laminated plate with flexibly clamped edges.

There have been few investigations in wave propagation in plates at large amplitudes. Sun, Gilmore and Koh[8] studied the influence of large amplitude on propagation of free waves in a laminated beam using a microstructure theory. Herrmann[9] investigated the flexural waves of large amplitudes in homogeneous elastic isotropic plates. The small number of papers on this subject is evidently due to the existence of nonlinear differential equations which do not allow a single harmonic wave to propagate.

The present paper deals with wave propagation in general laminates involving large deflections. Two cases are considered. In the first case, we are concerned with free waves of large amplitudes propagating in an initially flat plate. In the second case we look into the free waves of infinitesimal magnitudes superposed on a plate under large static deflection. The analysis in this work is simplified by using a special substitution. Due to the substitution, the equations of motion appear to be linear partial differential equations which, otherwise, would be nonlinear if expressed in terms of the plate displacement components.

2. THE NONLINEAR PLATE EQUATIONS

Consider a flat layer of fiber-reinforced material which can be regarded as an equivalent orthotropic medium. Let the layer lie in the x–y plane with the z-axis being perpendicular to it. The stress-equations of motion in Lagrangian description for large deformation can be expressed in the form[10]:

$$\frac{\partial}{\partial x_j}\left[S_{jk}\left(\delta_{ik}+\frac{\partial \bar{u}_i}{\partial x_k}\right)\right] = \rho \ddot{\bar{u}}_i \tag{1}$$

where x_i are the Lagrangian coordinates, $\bar{u}_i$ the displacement components, ρ the initial mass density, S_{jk} the components of the Kirchhoff stress tensor and δ_{ik} is the Kronecker delta. Following the assumptions of von Karman's large deflection theory of plates we retain in equation (1) the linear terms and only those product terms representing large transverse deflection, or large slopes given by $\partial\bar{w}/\partial x$ and $\partial\bar{w}/\partial y$. As a result, equation (1) becomes

$$\frac{\partial}{\partial x} S_{xx} + \frac{\partial}{\partial y} S_{yx} + \frac{\partial}{\partial z} S_{zx} = \rho \ddot{\bar{u}} \tag{2}$$

$$\frac{\partial}{\partial x} S_{xy} + \frac{\partial}{\partial y} S_{yy} + \frac{\partial}{\partial z} S_{zy} = \rho \ddot{\bar{v}} \tag{3}$$

$$\frac{\partial}{\partial x}\left(S_{xx}\frac{\partial \bar{w}}{\partial x} + S_{xy}\frac{\partial \bar{w}}{\partial y} + S_{xz}\right) + \frac{\partial}{\partial y}\left(S_{yx}\frac{\partial \bar{w}}{\partial x} + S_{yy}\frac{\partial \bar{w}}{\partial y} + S_{yz}\right)$$
$$+ \frac{\partial}{\partial z}\left(S_{zx}\frac{\partial \bar{w}}{\partial x} + S_{zy}\frac{\partial \bar{w}}{\partial y} + S_{zz}\right) = \rho \ddot{\bar{w}}. \tag{4}$$

We now consider a laminated plate of thickness h consisting of a finite number of layers of fiber-reinforced materials. Since we have assumed that the nonlinearity was geometrical in nature we will employ the usual linear relations between the stress and strain. We have for each layer

$$\begin{bmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} S_{yz} \\ S_{xz} \end{bmatrix} = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix} \begin{bmatrix} 2E_{yz} \\ 2E_{xz} \end{bmatrix} \tag{6}$$

where E_{xx}, E_{yy}, etc. are the Lagrangian strain components and Q_{ij} are the reduced stiffnesses which are related to the elastic constants by[11]

$$Q_{i\alpha} = c_{i\alpha} - \frac{c_{i3}}{c_{33}} c_{3\alpha} \qquad i = 1, 2, 3, 6, \alpha = 1, 2, 6 \tag{7}$$

where c_{ij} are the elastic constants. The relations given by equations (5 and 6) are obtained by assuming that the normal stress S_{zz} is negligible.

An approximate displacement field for the laminate which can account for the transverse shear deformation is given by

$$\begin{aligned}\bar{u} &= u(x, y) + z\psi_x(x, y)\\ \bar{v} &= v(x, y) + z\psi_y(x, y)\\ \bar{w} &= w(x, y)\end{aligned} \tag{8}$$

where u, v, w are the displacement components in the midplane of the laminate, ψ_x and ψ_y are the rotations of the cross-sections perpendicular to the x- and y-axes, respectively. Using the approximate displacement field, the Lagrangian strain components can be computed. If we again retain only the nonlinear terms representing large slopes, we obtain

$$\begin{aligned}E_{xx} &= \frac{\partial u}{\partial x} + z\frac{\partial \psi_x}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ E_{yy} &= \frac{\partial v}{\partial y} + z\frac{\partial \psi_y}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ 2E_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\\ 2E_{yz} &= \psi_y + \frac{\partial w}{\partial y}\\ 2E_{xz} &= \psi_x + \frac{\partial w}{\partial x}.\end{aligned} \tag{9}$$

After substituting equation (8) in the right-hand-sides of equations (2–4), the plate-stress-equations of motion can be derived in the usual manner by performing integrations over the thickness of the plate. We obtain

$$\begin{aligned}&\frac{\partial}{\partial x}N_x + \frac{\partial}{\partial y}N_{xy} + f_x = P\ddot{u} + R\ddot{\psi}_x\\ &\frac{\partial}{\partial x}N_{xy} + \frac{\partial}{\partial y}N_y + f_y = P\ddot{v} + R\ddot{\psi}_y\\ &\frac{\partial}{\partial x}M_x + \frac{\partial}{\partial y}M_{xy} - Q_x + m_x = R\ddot{u} + I\ddot{\psi}_x\\ &\frac{\partial}{\partial x}M_{xy} + \frac{\partial}{\partial y}M_y - Q_y + m_y = R\ddot{v} + I\ddot{\psi}_y\\ &\frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y} + Q_x\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_y\frac{\partial w}{\partial y} + Q_y\right) + q = p\ddot{w}\end{aligned} \tag{10}$$

where

$$\begin{aligned}&N_x = \int_{-h/2}^{h/2} S_{xx}\,\mathrm{d}z,\ N_{xy} = \int_{-h/2}^{h/2} S_{xy}\,\mathrm{d}z,\ N_y = \int_{-h/2}^{h/2} S_{yy}\,\mathrm{d}z\\ &Q_x = \int_{-h/2}^{h/2} S_{xz}\,\mathrm{d}z,\ Q_y = \int_{-h/2}^{h/2} S_{yz}\,\mathrm{d}z\end{aligned} \tag{11}$$

are the stress resultants;

$$M_x = \int_{-h/2}^{h/2} S_{xx} z \, dz, \; M_{xy} = \int_{-h/2}^{h/2} S_{xy} z \, dz, \; M_y = \int_{-h/2}^{h/2} S_{yy} z \, dz \tag{12}$$

are the moments; f_x and f_y are the tangential loadings; m_x and m_y are the external moments; q is the transverse load; and

$$(P, R, I) = \int_{-h/2}^{h/2} \rho(1, z, z^2) \, dz. \tag{13}$$

It should be noted that in deriving equation (10), integrals involving S_{zz} have been neglected.

3. FREE WAVE PROPAGATION AT LARGE AMPLITUDE

We consider a train of plane free waves propagating in the η-direction which makes an angle θ with the x-axis. The variable η is given by

$$\eta = x \cos\theta + y \sin\theta. \tag{14}$$

All the field quantities are now assumed to be functions of η and time only. Using the relation given by equation (14), we can write the plate equations of motion given by equation (10) as

$$\begin{aligned}
&\frac{\partial}{\partial\eta}(\alpha N_x + \beta N_{xy}) = P\ddot{u} + R\ddot{\psi}_x \\
&\frac{\partial}{\partial\eta}(\alpha N_{xy} + \beta N_y) = P\ddot{v} + R\ddot{\psi}_y \\
&\frac{\partial}{\partial\eta}(\alpha M_x + \beta M_{xy}) - Q_x = R\ddot{u} + I\ddot{\psi}_x \\
&\frac{\partial}{\partial\eta}(\alpha M_{xy} + \beta M_y) - Q_y = R\ddot{v} + I\ddot{\psi}_y \\
&\frac{\partial}{\partial\eta}\left\{(\alpha^2 N_x + 2\alpha\beta N_{xy} + \beta^2 N_y)\frac{\partial w}{\partial\eta} + \alpha Q_x + \beta Q_y\right\} = P\ddot{w}
\end{aligned} \tag{15}$$

where $\alpha = \cos\theta$, $\beta = \sin\theta$ and the external loads are assumed to be absent.

The strain components become

$$\begin{aligned}
E_{xx} &= \alpha\left(g + z\frac{\partial\psi_x}{\partial\eta}\right) \\
E_{yy} &= \beta\left(f + z\frac{\partial\psi_y}{\partial\eta}\right) \\
2E_{xy} &= \beta\left(g + z\frac{\partial\psi_x}{\partial\eta}\right) + \alpha\left(f + z\frac{\partial\psi_y}{\partial\eta}\right) \\
2E_{yz} &= \psi_y + \beta\frac{\partial w}{\partial\eta} \\
2E_{xz} &= \psi_x + \alpha\frac{\partial w}{\partial\eta}
\end{aligned} \tag{16}$$

where

$$g = \frac{\partial u}{\partial \eta} + \frac{1}{2}\alpha \left(\frac{\partial w}{\partial \eta}\right)^2$$

$$f = \frac{\partial v}{\partial \eta} + \frac{1}{2}\beta \left(\frac{\partial w}{\partial \eta}\right)^2. \tag{17}$$

Substituting equation (16) in equations (5 and 6) we obtain

$$\begin{bmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \end{bmatrix} = [\bar{Q}][T]\begin{bmatrix} g \\ f \end{bmatrix} + z[\bar{Q}][T]\begin{bmatrix} \dfrac{\partial \psi_x}{\partial \eta} \\ \dfrac{\partial \psi_y}{\partial \eta} \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} S_{yz} \\ S_{xz} \end{bmatrix} = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix}\begin{bmatrix} \psi_y + \beta \dfrac{\partial w}{\partial \eta} \\ \psi_x + \alpha \dfrac{\partial w}{\partial \eta} \end{bmatrix}$$

where the matrices $[\bar{Q}]$ and $[T]$ are given by

$$[\bar{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \tag{20}$$

$$[T] = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \\ \beta & \alpha \end{bmatrix}. \tag{21}$$

Employing the conventional notations

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2)\,dz \tag{22}$$

the plate–stress resultants and moments can be expressed as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = [\bar{A}][T]\begin{bmatrix} g \\ f \end{bmatrix} + [\bar{B}][T]\begin{bmatrix} \dfrac{\partial \psi_x}{\partial \eta} \\ \dfrac{\partial \psi_y}{\partial \eta} \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = [\bar{B}][T]\begin{bmatrix} g \\ f \end{bmatrix} + [\bar{D}][T]\begin{bmatrix} \dfrac{\partial \psi_x}{\partial \eta} \\ \dfrac{\partial \psi_y}{\partial \eta} \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix}\begin{bmatrix} \psi_y + \beta \dfrac{\partial w}{\partial \eta} \\ \psi_x + \alpha \dfrac{\partial w}{\partial \eta} \end{bmatrix}. \tag{25}$$

In equations (23 and 24)

$$[\bar{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} = \int_{-h/2}^{h/2} [\bar{Q}]\, dz \tag{26}$$

and

$$([\bar{B}], [\bar{D}]) = \int_{-h/2}^{h/2} (z, z^2)[\bar{Q}]\, dz. \tag{27}$$

It is known that in a homogeneous and isotropic plate undergoing large amplitude transverse motion, the effects of the terms $\ddot{u}$ and $\ddot{v}$ are secondary[12]. Due to the heterogeneous property over the thickness of plate, there is strong coupling between the flexural and extensional motions in the laminated plate. Although the coupling can be very strong, the extensional inertia force still has no significant effect on the flexural motion of the plate. In the Appendix, the influence of the inertia force of the extensional motion induced by the flexural–extensional coupling is investigated for the case of small deflection. It is found that the neglect of the inertias does not give rise to any significant errors.

In view of the foregoing, we now neglect the terms $\ddot{u}$ and $\ddot{v}$ in equation (15). Since we are interested in the laminated plate consisting of layers of identical mass densities, we also have $R = 0$. Thus, the equations of motion become

$$\begin{aligned} &\frac{\partial}{\partial \eta}(\alpha N_x + \beta N_{xy}) = 0 \\ &\frac{\partial}{\partial \eta}(\alpha N_{xy} + \beta N_y) = 0 \\ &\frac{\partial}{\partial \eta}(\alpha M_x + \beta M_{xy}) - Q_x = I\ddot{\psi}_x \\ &\frac{\partial}{\partial \eta}(\alpha M_{xy} + \beta M_y) - Q_y = I\ddot{\psi}_y \\ &\frac{\partial}{\partial \eta}\left\{(\alpha K_1 + \beta K_2)\frac{\partial w}{\partial \eta} + \alpha Q_x + \beta Q_y\right\} = P\ddot{w} \end{aligned} \tag{28}$$

where

$$\begin{aligned} K_1 &= \alpha N_x + \beta N_{xy} \\ K_2 &= \alpha N_{xy} + \beta N_y \end{aligned} \tag{29}$$

are two constants (or functions of time only) obtained from integrating the first two equations in equation (28).

The displacement-equations of motion can be derived by substituting equations (23–25) in equation (28). Defining the matrices

$$\begin{aligned} [A^*] &= [T]^T[\bar{A}][T] \\ [B^*] &= [T]^T[\bar{B}][T] \\ [D^*] &= [T]^T[\bar{D}][T] \end{aligned} \tag{30}$$

we can put the displacement-equations of motion in the form

$$
\begin{aligned}
&A^*_{11}\frac{\partial g}{\partial \eta}+A^*_{12}\frac{\partial f}{\partial \eta}+B^*_{11}\frac{\partial^2\psi_x}{\partial \eta^2}+B^*_{12}\frac{\partial^2\psi_y}{\partial \eta^2}=0\\
&A^*_{21}\frac{\partial g}{\partial \eta}+A^*_{22}\frac{\partial f}{\partial \eta}+B^*_{21}\frac{\partial^2\psi_x}{\partial \eta^2}+B^*_{22}\frac{\partial^2\psi_y}{\partial \eta^2}=0\\
&B^*_{11}\frac{\partial g}{\partial \eta}+B^*_{12}\frac{\partial f}{\partial \eta}+D^*_{11}\frac{\partial^2\psi_x}{\partial \eta^2}+D^*_{12}\frac{\partial^2\psi_y}{\partial \eta^2}-A_{55}\psi_x-A_{45}\psi_y\\
&\qquad-(\beta A_{45}+\alpha A_{55})\frac{\partial w}{\partial \eta}=I\ddot{\psi}_x\\
&B^*_{21}\frac{\partial g}{\partial \eta}+B^*_{22}\frac{\partial f}{\partial \eta}+D^*_{21}\frac{\partial^2\psi_x}{\partial \eta^2}+D^*_{22}\frac{\partial^2\psi_y}{\partial \eta^2}-A_{45}\psi_x-A_{44}\psi_y\\
&\qquad-(\beta A_{44}+\alpha A_{45})\frac{\partial w}{\partial \eta}=I\ddot{\psi}_y\\
&(\alpha K_1+\beta K_2)\frac{\partial^2 w}{\partial \eta^2}+(\alpha A_{55}+\beta A_{45})\frac{\partial \psi_x}{\partial \eta}+(\alpha A_{45}+\beta A_{44})\frac{\partial \psi_y}{\partial \eta}\\
&\qquad+(\alpha^2A_{55}+2\alpha\beta A_{45}+\beta^2A_{44})\frac{\partial^2 w}{\partial \eta^2}=P\ddot{w}.
\end{aligned}
\tag{31}
$$

It should be noted that the equations of motion given by (31) are nonlinear partial differential equations in u, v, w, $\ddot{\psi}_x$ and ψ_y. A single harmonic wave thus can not satisfy the equations. However, in the present form which is expressed in terms of g, f, w, ψ_x and ψ_y, the equations appear to be linear partial differential equations. Taking advantage of this linearity, we can assume that a free wave is given by

$$
\begin{aligned}
w&=W\cos k(\eta-ct)\\
\psi_x&=\Psi_x\sin k(\eta-ct)\\
\psi_y&=\Psi_y\sin k(\eta-ct)\\
g&=G\cos k(\eta-ct)+\tfrac{1}{4}\alpha W^2k^2\\
f&=F\cos k(\eta-ct)+\tfrac{1}{4}\beta W^2k^2
\end{aligned}
\tag{32}
$$

where W, Ψ_x, Ψ_y, G and F are constants, k is the angular wave number and c the phase velocity for the transverse motion. The constant terms in the last two equations in equation (32) are added to assure that u and v are harmonic in η.

Using the expressions of g and f as given by equation (32) together with equation (17) we obtain

$$
\begin{aligned}
u&=\frac{G}{k}\sin k(\eta-ct)+\frac{\alpha W^2k}{8}\sin 2k(\eta-ct)\\
v&=\frac{F}{k}\sin k(n-ct)+\frac{\beta W^2k}{8}\sin 2k(\eta-ct).
\end{aligned}
\tag{33}
$$

Equation (33) reveals that there exist two trains of extensional harmonic waves. It is also evident from equation (33) that in the linear case where $W \ll 1$, the second harmonic wave can be neglected.

The two constants K_1 and K_2 given by equation (29) can now be determined by substituting equation (32) in equation (23) and subsequently in equation (29). We obtain

$$\begin{aligned} K_1 &= \tfrac{1}{4}W^2k^2(\alpha A_{11}^* + \beta A_{12}^*) \\ K_2 &= \tfrac{1}{4}W^2k^2(\alpha A_{21}^* + \beta A_{22}^*). \end{aligned} \tag{34}$$

It must be observed that in deriving equation (34), the conditions $\partial K_1/\partial\eta = 0$ and $\partial K_2/\partial\eta = 0$ have been employed.

Substitution of equation (32) in equation (31) leads to a system of five homogeneous equations. A nontrivial solution exists if the determinant of coefficients vanishes, i.e.

$$\begin{vmatrix} A_{11}^* & A_{12}^* & B_{11}^*k & B_{12}^*k & 0 \\ A_{21}^* & A_{22}^* & B_{21}^*k & B_{22}^*k & 0 \\ B_{11}^*k & B_{12}^*k & (D_{11}^* - Ic^2)k^2 + A_{55} & D_{12}^*k^2 + A_{45} & \alpha A_{55} + \beta A_{45} \\ B_{21}^*k & B_{22}^*k & D_{21}^*k^2 + A_{45} & (D_{22}^* - Ic^2)k^2 + A_{44} & \beta A_{44} + \alpha A_{45} \\ 0 & 0 & \alpha A_{55} + \beta A_{45} & \beta A_{44} + \alpha A_{45} & \begin{array}{l} \alpha K_1 + \beta K_2 + \alpha^2 A_{55} \\ + 2\alpha\beta A_{45} + \beta^2 A_{44} \\ - Pc^2 \end{array} \end{vmatrix} = 0. \tag{35}$$

For cross-ply laminates the dispersion equation (35) can be simplified if the wave propagates in the x-direction and a state of plane strain parallel to y–z plane exists. In this case we have

$$\alpha = 1, \qquad \beta = 0, \qquad v = 0, \qquad \psi_y = 0 \tag{36}$$

and the dispersion equation becomes

$$\begin{vmatrix} A_{11}^* & B_{11}^*k & 0 \\ B_{11}^*k & (D_{11}^* - Ic^2)k^2 + A_{55} & A_{55} \\ 0 & A_{55} & (K_1 + A_{55} - Pc^2) \end{vmatrix} = 0. \tag{37}$$

If, in addition, we neglect the rotatory inertia and noting that $A_{11}^* = A_{11}$, $B_{11}^* = B_{11}$, $D_{11}^* = D_{11}$ then we obtain the phase velocity for the transverse wave as

$$Pc^2 = \frac{1}{4}A_{11}W^2k^2 + \frac{A_{55}k^2(A_{11}D_{11} - B_{11}^2)}{k^2(A_{11}D_{11} - B_{11}^2) + A_{11}A_{55}}. \tag{38}$$

4. FREE WAVES OF SMALL AMPLITUDE SUPERPOSED ON LARGE DEFLECTION

It is of interest to investigate the dynamic response of an initially deformed plate. The behavior of the incremental deformation in a deformed laminated plate should be quite different from that of the initially stress free plate even though the incremental deformation is infinitesimal in nature. It is expected that the incremental plate-stress–displacement relations are not independent of the initial deflection.

Let the initial deflection of the laminated plate be denoted by u_0, v_0, w_0, ψ_{x0} and ψ_{y0}, and the incremental deformation by u', v', w', ψ_x' and ψ_y'. The final state of deflection is then given by

$$\begin{aligned} u &= u_0 + u' \\ v &= v_0 + v' \\ w &= w_0 + w' \\ \psi_x &= \psi_{x0} + \psi'_x \\ \psi_y &= \psi_{y0} + \psi'_y . \end{aligned} \tag{39}$$

The plate-stress resultants and moments are expressed in a similar manner, i.e.

$$N_x = N_{x0} + N'_x, \qquad N_{xy} = N_{xy0} + N'_{xy}, \text{ etc.} \tag{40}$$

It is assumed that the initial deflection is in a state of static equilibrium and depends only on η. Thus, the initial stress-resultants and moments satisfy the equations of equilibrium which can be obtained from equation (15) by dropping the inertia terms.

Consider an incremental deformation which varies only in the η-direction. Substituting equations (39) and (40) in equation (15) and making use of the condition that the initial deformation is in equilibrium, we obtain the incremental-stress-equations of motion as

$$\begin{aligned} &\frac{\partial}{\partial \eta}(\alpha N'_x + \beta N'_{xy}) = P\ddot{u}' + R\ddot{\psi}'_x \\ &\frac{\partial}{\partial \eta}(\alpha N'_{xy} + \beta N'_y) = P\ddot{v}' + R\ddot{\psi}'_y \\ &\frac{\partial}{\partial \eta}(\alpha M'_x + \beta M'_{xy}) - Q'_x = R\ddot{u}' + I\ddot{\psi}'_x \\ &\frac{\partial}{\partial \eta}(\alpha M'_{xy} + \beta M'_y) - Q'_y = R\ddot{v}' + I\ddot{\psi}'_y \\ &\frac{\partial}{\partial \eta}\left\{(\alpha K_1^0 + \beta K_2^0)\frac{\partial w'}{\partial \eta} + (\alpha^2 N'_x + 2\alpha\beta N'_{xy} + \beta^2 N'_y)\frac{\partial w_0}{\partial \eta} \right. \\ &\qquad \left. + \alpha Q'_x + \beta Q'_y\right\} = P\ddot{w}' \end{aligned} \tag{41}$$

where

$$\begin{aligned} K_1^0 &= \alpha N_{x0} + \beta N_{xy0} \\ K_2^0 &= \alpha N_{xy0} + \beta N_{y0} . \end{aligned} \tag{42}$$

The incremental plate-stress–strain relations can be obtained by substituting equations (39 and 40) in equations (23–25). It is found that the incremental relations are formally the same as those given by equations (23–25) with ψ_x, ψ_y, w, g and f being replaced by ψ'_x, ψ'_y, w',

$$g' = \frac{\partial u'}{\partial \eta} + \alpha \frac{\partial w_0}{\partial \eta}\frac{\partial w'}{\partial \eta} \tag{43}$$

and

$$f' = \frac{\partial v'}{\partial \eta} + \beta \frac{\partial w_0}{\partial \eta}\frac{\partial w'}{\partial \eta} \tag{44}$$

respectively. It should be noted that in deriving equations (43 and 44) the higher order terms in $\partial w'/\partial\eta$ have been neglected.

Consider the initial deflection described by

$$w_0 = W_0 \cos k_0 \eta. \tag{45}$$

The other initial displacement components u_0, v_0, ψ_{x0} and ψ_{y0} can be accordingly derived from the equations of equilibrium. However, it should be noted that a static transverse load must be added in order to maintain the desired deflection given by equation (45). The two constants K_1^0 and K_2^0 now have the expressions

$$\begin{aligned} K_1^0 &= \tfrac{1}{4} W_0^2 k_0^2 (\alpha A_{11}^* + \beta A_{12}^*) \\ K_2^0 &= \tfrac{1}{4} W_0^2 k_0^2 (\alpha A_{21}^* + \beta A_{22}^*) \end{aligned} \tag{46}$$

which are obtained from equation (34).

Based upon the same argument as given in the previous section, it is again assumed that

$$R = 0, \qquad \ddot{u}' = 0, \qquad \ddot{v}' = 0. \tag{47}$$

As a consequence, from the first two equations in equation (41) we can obtain

$$\begin{aligned} \alpha N_x' + \beta N_{xy}' &= K_3 \\ \alpha N_{xy}' + \beta N_y' &= K_4 \end{aligned} \tag{48}$$

where K_3 and K_4 are two constants.

Consider a time–harmonic wave propagating in an initially deflected laminate described by equation (45). The incremental displacements are given by

$$\begin{aligned} w' &= W' \cos k(\eta - ct) \\ \psi_x' &= \Psi_x' \sin k(\eta - ct) \\ \psi_y' &= \Psi_y' \sin k(\eta - ct) \\ g' &= G' \cos k(\eta - ct) \\ f' &= F' \cos k(\eta - ct) \end{aligned} \tag{49}$$

where W', Ψ_x', Ψ_y', G' and F' are constants. The corresponding expressions for u' and v' can be obtained from integrating equations (43 and 44), respectively. For $k \neq k_0$, we obtain

$$\begin{aligned} u' &= \frac{1}{k} G' \sin k(\eta - ct) - \frac{1}{2} \alpha k_0 k W_0 W' \left\{ \frac{1}{k - k_0} \sin\left[(k - k_0)\eta - kct\right] \right. \\ &\quad \left. - \frac{1}{k + k_0} \sin\left[(k + k_0)\eta - kct\right] \right\} \\ v' &= \frac{1}{k} F' \sin k(\eta - ct) - \frac{1}{2} \beta k_0 k W_0 W' \left\{ \frac{1}{k - k_0} \sin\left[(k - k_0)\eta - kct\right] \right. \\ &\quad \left. - \frac{1}{k + k_0} \sin\left[(k + k_0)\eta - kct\right] \right\}. \end{aligned} \tag{50}$$

Thus, the extensional motion is composed of three harmonic waves propagating with three different phase velocities, namely, c, $ck/k - k_0$, and $ck/k + k_0$ with the corresponding wave numbers k, $k - k_0$, and $k + k_0$.

If $k = k_0$, it can be easily shown that

$$\begin{aligned} u' &= \frac{1}{k} G' \sin k_0(\eta - ct) + \frac{1}{4} \alpha k_0 W_0 W' \sin k_0(2\eta - ct) - \frac{\eta}{2} \alpha k_0^2 W_0 W' \cos \omega_0 t \\ v' &= \frac{1}{k} F' \sin k_0(\eta - ct) + \frac{1}{4} \beta k_0 W_0 W' \sin k_0(2\eta - ct) - \frac{\eta}{2} \beta k_0^2 W_0 W' \cos \omega_0 t \end{aligned} \tag{51}$$

where

$$\omega_0 = k_0 c \tag{52}$$

is the angular frequency. It is noted that the solutions given by equation (51) blow up as $\eta \to \infty$. The situation can be eliminated by adding the terms

$$\tfrac{1}{2} \alpha k_0^2 W_0 W' \cos \omega_0 t \tag{53}$$

and

$$\tfrac{1}{2} \beta k_0^2 W_0 W' \cos \omega_0 t \tag{54}$$

in the last two equations of equation (49), respectively.

Using the same procedure for evaluating K_1 and K_2 presented in the previous section, we can easily show that

$$K_3 = K_4 = 0. \tag{55}$$

In view of equations (47 and 55), equation (41) can be reduced to

$$\begin{aligned} &\frac{\partial}{\partial \eta} (\alpha N'_x + \beta N'_{xy}) = 0 \\ &\frac{\partial}{\partial \eta} (\alpha N'_{xy} + \beta N'_y) = 0 \\ &\frac{\partial}{\partial \eta} (\alpha M'_x + \beta M'_{xy}) - Q'_x = I \ddot{\psi}'_x \\ &\frac{\partial}{\partial \eta} (\alpha M'_{xy} + \beta M'_y) - Q'_y = I \ddot{\psi}'_y \\ &\frac{\partial}{\partial \eta} \left\{ (\alpha K_1^0 + \beta K_2^0) \frac{\partial w'}{\partial \eta} + \alpha Q'_x + \beta Q'_y \right\} = P \ddot{w}'. \end{aligned} \tag{56}$$

A comparison of equation (56) with equation (28) shows that the equations of motion for the incremental wave motion have the same form as the large amplitude wave except now K_1 and K_2 should be replaced by K_1^0 and K_2^0, respectively. Furthermore, due to the similar plate-stress–strain relations, the incremental displacement-equations of motion are given by equation (31) with g, f, ψ_x, ψ_y and w being replaced by g', f', ψ'_x, ψ'_y and w', respectively. As a consequence, the dispersion equation for the incremental time-harmonic wave can be obtained from equation (35) by replacing K_1 and K_2 with K_1^0 and K_2^0, respectively.

The phase velocity for the special case where $\alpha = 1$, $\beta = 0$ can be easily computed. If the rotatory inertias are neglected, then the phase velocity for a free wave propagating in a cross-ply laminate is

$$Pc^2 = \frac{1}{4} A_{11} W_0^2 k_0^2 + \frac{A_{55} k^2 (A_{11} D_{11} - B_{11}^2)}{k^2 (A_{11} D_{11} - B_{11}^2) + A_{11} A_{55}}. \tag{57}$$

The limiting phase velocity for infinitely long wavelength ($k \to 0$) is obtained as

$$Pc^2 = \tfrac{1}{4} A_{11} W_0^2 k_0^2 \tag{58}$$

It becomes obvious that the limiting phase velocity is in direct proportion to the initial deflection and its wave number.

5. NUMERICAL RESULTS

In order to perform numerical investigations of the wave propagations we consider a typical graphite-epoxy composite whose elastic properties are given by the following engineering constants:

$$\begin{aligned} &E_L = 25 \times 10^6 \text{ psi}, \quad &&E_T = 1 \times 10^6 \text{ psi}, \quad &&G_{LT} = 0{\cdot}5 \times 10^6 \text{ psi} \\ &G_{TT} = 0{\cdot}2 \times 10^6 \text{ psi}, \quad &&\nu_{LT} = 0{\cdot}25 \quad &&\nu_{TT} = 0{\cdot}25 \end{aligned} \tag{59}$$

where L and T are the directions parallel and normal to the fibers, respectively, ν_{LT} is the Poisson's ratio measuring lateral strain under uniaxial normal stress parallel to the fibers, and ν_{TT} is the Poisson's ratio defined in the same manner. It is also assumed that the fibrous material of each layer possesses the square-symmetry. The nonvanishing elastic constants c_{ij} for a 0-degree layer (i.e. the fibers are parallel to the x-axis) can be obtained as

$$\begin{aligned} &c_{11} = 25{\cdot}167 \times 10^6 \text{ psi}, \quad &&c_{12} = c_{21} = c_{13} = c_{31} = 0{\cdot}335 \times 10^6 \text{ psi} \\ &c_{22} = 1{\cdot}071 \times 10^6 \text{ psi}, \quad &&c_{23} = c_{32} = 0{\cdot}271 \times 10^6 \text{ psi} \\ &c_{33} = 1{\cdot}071 \times 10^6 \text{ psi}, \quad &&c_{44} = 0{\cdot}2 \times 10^6 \text{ psi} \\ &c_{55} = c_{66} = 0{\cdot}5 \times 10^6 \text{ psi}. && \end{aligned} \tag{60}$$

The reduced stiffness coefficients in equations (5 and 6) can be derived from equation (7). We obtain

$$\begin{aligned} &Q_{11} = 25{\cdot}062 \times 10^6 \text{ psi}, \quad &&Q_{12} = 0{\cdot}250 \times 10^6 \text{ psi} \\ &Q_{22} = 1{\cdot}002 \times 10^6 \text{ psi}, \quad &&Q_{66} = 0{\cdot}5 \times 10^6 \text{ psi} \\ &Q_{44} = 0{\cdot}2 \times 10^6 \text{ psi}, \quad &&Q_{55} = 0{\cdot}5 \times 10^6 \text{ psi} \\ &Q_{16} = Q_{26} = Q_{45} = 0. && \end{aligned} \tag{61}$$

The reduced stiffness coefficients for layers with fibers orienting in other directions can be obtained by the usual coordinate transformation law.

Using the values given by equation (61) we can evaluate numerically the dispersion equations. Fig. 1 shows the phase velocity for a transverse wave with large amplitudes propagating in the x-direction in a 2-layered (0°, 90°) and 3-layered (0°, 90°, 0°) cross-ply laminates. The stiffening effect of the large deflection can be seen to be very substantial. In order to find the influence of anisotropy of the composite material, we present in Fig. 2 the phase velocity vs the direction of wave propagation at $kh = 1$. Two values of amplitude are considered, i.e. $W/h = 1{\cdot}0$ and $0{\cdot}5$.

The dimensionless phase velocities for free waves of infinitesimal amplitudes propagating in the x-direction in deformed laminates are shown in Fig. 3. The initial deflections are given by equation (45) with $W_0/h = 1{\cdot}0, 0{\cdot}5, 0{\cdot}1$. It is seen that the initial deformation would increase the phase velocity as well as make it less dispersive. For $k_0 h = 0{\cdot}5$, $kh = 1{\cdot}0$ and $W_0/h = 0{\cdot}5$, $1{\cdot}0$ the dimensionless phase velocity is plotted against the direction of wave propagation. In Fig. 4 the variation of the phase velocity with respect to the direction of propagation is shown.

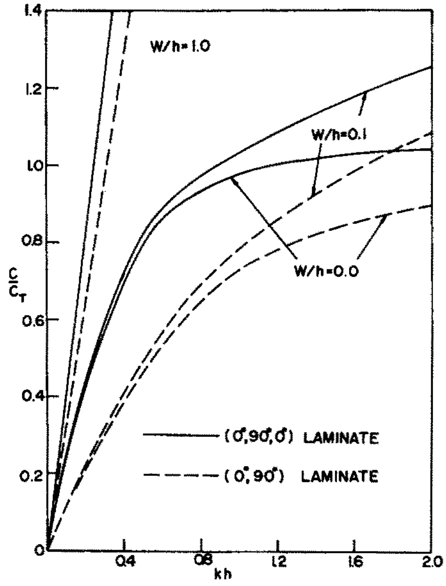

Fig. 1. Phase velocity for the large amplitude wave propagating in the x-direction, $c_T^2 = (G_{LT} + G_{TT})h/2P$.

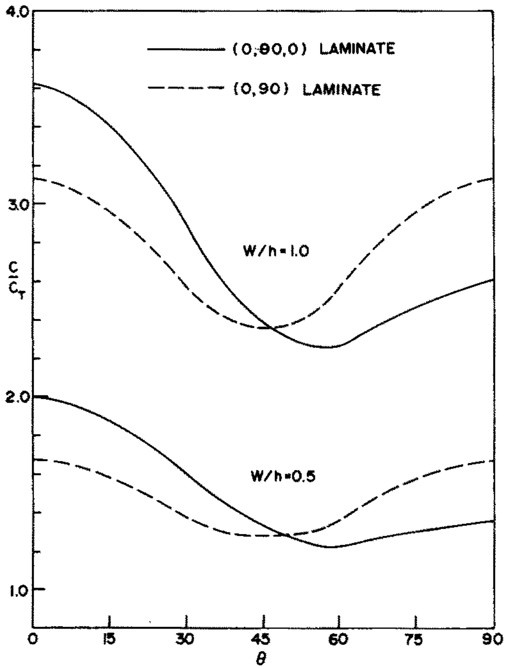

Fig. 2. Variation of the phase velocity for the large amplitude wave against direction of propagation at $kh = 1$, $c_T^2 = (G_{LT} + G_{TT})h/2P$.

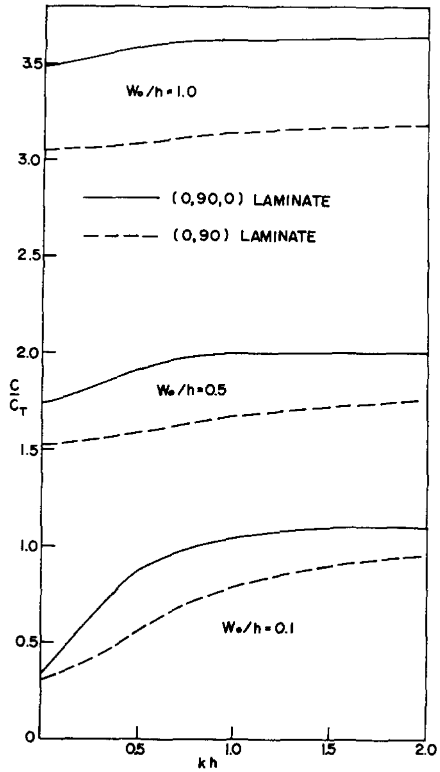

Fig. 3. Influence of initial deformation on the incremental wave propagating in the x-direction, $c_T^2 = (G_{LT} + G_{TT})h/2P$.

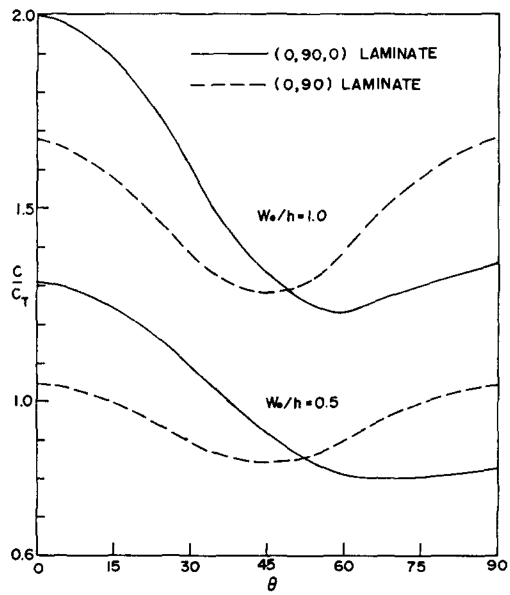

Fig. 4. Phase velocity for the incremental wave vs. angle of propagation for $kh = 0{\cdot}5$, $kh = 1$, $c_T^2 = (G_{LT} + G_{TT})h/2P$.

6. CONCLUSIONS

Equations governing free wave propagation at large amplitudes in general laminates of composite materials have been derived. The transverse shear deformation as well as the rotatory inertia were included. The equations for incremental motions superposed on a deformed laminate in static large deflection were also presented. Using a substitution for the extensional displacements the equations of motion were expressed as linear partial differential equations and the investigation of free wave propagation was simplified drastically. It was found that large deflections have substantial stiffening effect on the phase velocity. Due to the simplified method used, we were able to find that a combination of a finite number of harmonic waves could propagate in a flat plate at large amplitudes and in a deformed plate at small amplitudes.

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APPENDIX

In order to determine the influence of the longitudinal as well as the rotatory inertias on the phase velocity, we consider the linear case. A two-layered cross-ply laminate which exhibits the most severe bending–extension coupling is taken as the example. The direction of propagation of the harmonic wave is taken to be 15° measured from the x-axis. Four cases are investigated. In Case 1 both longitudinal and rotatory inertias are neglected; in Case 2 and Case 3 the rotatory inertia and the longitudinal inertia are neglected, respectively; and Case 4 includes all the inertia terms. The dispersion equations corresponding to the four cases can be obtained easily, and will not be reproduced here. The numerical results are shown in the table below with the dimensionless phase velocity $c/c_T(c_T^2 = h(G_{LT} + G_{TT})/2P)$ against the dimensionless wave number kh. The material constants given by equation (59) are used.

Table 1

kh \ $\frac{c}{c_T}$	Case 1	Case 2	Case 3	Case 4
0·0	0·0	0·0	0·0	0·0
0·2	0·1971	0·1969	0·1968	0·1966
0·4	0·3725	0·3713	0·3707	0·3695
0·6	0·5147	0·5119	0·5105	0·5078
0·8	0·6235	0·6194	0·6173	0·6131
1·0	0·7046	0·6995	0·6969	0·6919
1·2	0·7646	0·7592	0·7563	0·7507
2·0	0·8896	0·8848	0·8819	0·8765
3·0	0·9449	0·9416	0·9352	0·9352